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Extended Complex Trigonometry in Relation to Integrable $2D-$Quantum Field Theories and Duality

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Abstract

Multicomplex numbers of order n have an associated trigonometry (multisine functions with $n-1$ parameters) leading to a natural extension of the sine-Gordon model. The parameters are constrained from the requirement of local current conservation. In two dimensions for $n < 6$ known integrable models (deformed Toda and non-linear sigma, pure affine Toda...) with dual counterparts are obtained in this way from the multicomplex space $\mathbb{MC}_n$ itself and from the natural embedding $\mathbb{MC}_n \subset \mathbb{MC}_m, n < m$. For $n \geq 6$ a generic constraint on the space of parametersis obtained from current conservation at first order in the interaction Lagragien.

1 Introduction

In the description of strongly interacting systems most of the relevant physical quantities such as spectra, correlation functions ..., cannot be obtained by perturbative methods and many techniques have been investigated to go beyond the regime of weak coupling. In gauge theories the electric-magnetic duality has been pointed out [1, 2] as a possibility to relate weak and strong coupling situations. It is by now common to speak of duality only in this sense and a well known example in two dimensions is the link between the sine-Gordon and massive Thirring models [3]. In a more recent past, in order to understand further the mechanisms of duality in mass spectra and $S-$matrices, the attention focussed on Toda systems. They are examples of integrable though deformed conformal field theories with affine Lie algebras symmetries showing exact duality [5]. An ever growing number of such deformed Toda models is currently proposed [6]. The question naturally arises of the existence of an underlying common structure suitable for a general analysis. The issue was studied in a recent publication [7]. This structure comes from the introduction of Clifford algebras of polynomial and generalized Clifford algebras [8]. These algebras are associated to the linearization of polynomials of degree n, in the same way the usual Clifford algebra is associated to Dirac's linearization of the laplacian operator. These Clifford families of order n contain a natural extension of complex numbers and trigonometric functions, called respectively multicomplex numbers and multisine-functions [9]. A generalization of the sine-Gordon model in the multicomplex space $\mathbb{MC}_n$ was proposed in [7]. A first investigation of integrability properties in two space-time dimensions was also given. In this procedure a necessary step is the construction and analysis of conserved quantities. Because of their evident link with integrable cases only low n cases ($n \leq 4$) were then studied. Here, new interesting properties of the multisine functions are exhibited which lead to a more general analysis. On the one hand a natural embedding of the multicomplex space exists which leads to "non-minimal" multisine-functions as opposed to the "minimal" ones considered in [7]. New multisine-Gordon models emerge which can be either identified to known integrable cases with dual counterparts or are truly new. On the other hand the multisine-functions can be written in a generic way which gives access to hidden symmetries and permits the disentanglement of the elaborated algebra associated with non-local conserved charges [10].

The purpose of this letter is to generalize the analysis of [7]. We shall leave apart most technicalities, but focus on some aspects in relation to duality and current conservation for arbitrary dimension of the multicomplex space $\mathbb{MC}_n$.

To start, a brief summary of the definition of the multisine-functions is given. They naturally contain $n-1$ parameters and their properties are given in relation to the different underlying possible structures of the MC-numbers. The conservation of *local* currents implies relations between the parameters. For real parameters specific solutions are identified to known integrable QFT's and their dual representations are exhibited. Some cases with imaginary parameters have original dual properties. An example is discussed for $n = 4$. Perspectives and conclusions are then drawn.

2 Multicomplex numbers and multisine functions

Multicomplex (MC) numbers [9] have been introduced in the past in relation to generalized Clifford algebra sustained by linearization of polynomials of degree higher than two [8]. The starting point is the introduction of a generator e, such that $e^n = -1$, sustaining the space of multicomplex numbers of order n, $\mathbb{MC}_n$. In keeping with the case $n = 2$ of usual complex numbers and their trigonometric functions, an associated extended trigonometry follows. It is characterized by specific "angular" functions dubbed multisine (mus). These mus-functions depend on $\phi_a, a = 0 \cdots, E(n/2) - 1$ compact quantities, and φ_a $a = 0, \cdots, n - E(n/2) - 1$ non-compact ones ($E(a)$ is the integer part of a). A collection of useful relations [9] exists between the mus-functions: additions, derivatives etc....
At variance with the quadratic case, where the sine and cosine functions are unique, when $n > 2$, different type of functions may be obtained. However when obtained as for the case $n = 2$ from the exponentiation of MC-numbers [7, 9, 10, 11] , all these mus-functions can be written under a common form. They are connected to the existence of natural embeddings of the multicomplex space $\mathbb{MC}_n \subset \mathbb{MC}_m, n < m$, with any n for m odd and even n when m even. Indeed, starting from $e_m, ((e_m)^m = -1)$, the generator of $\mathbb{MC}_m$, it is possible to define several generators of $\mathbb{MC}_n, e_{(n|m)}$, satisfying $(e_{(n|m)})^n = -1$ [11]. But now, if the algebra $\mathbb{MC}_{(n|m)}$ is considered as a sub-algebra of $\mathbb{MC}_m$, the associated mus-functions satisfy an $m-$th order relation instead of an $n-$th order one. In orther word, these mus-functions of order n are related to those of higher order m. They are given as

$$\left\{ \begin{array}{lll} \mathrm{mus}_k\left(\{\phi_a\},\{\varphi_a\}\right) = & \frac{2}{n}\sum\limits_{a=0}^{n/2-1}\cos(\phi_a - (2a+1)\frac{k\pi}{n})\exp(\varphi_a) & \text{for } n \text{ even} \\ \mathrm{mus}_k\left(\{\phi_a\},\{\varphi_a\}\right) = & \frac{2}{n}\sum\limits_{a=0}^{(n-1)/2-1}\cos(\phi_a - (2a+1)\frac{k\pi}{n})\exp(\varphi_a) + & \\ & \frac{(-1)^k}{n}\exp\left(\varphi_{(n-1)/2}\right) & \text{for } n \text{ odd.} \end{array} \right. \tag{2.1}$$

These mus-functions differ from each other only through the constraint necessary to ensure unimodular MC-numbers (cf $\|z\| = 1$ for $n = 2$) and which connects the non-compact variables. The φ_a's must verify

$$\left\{ \begin{array}{ll} \sum\limits_{a=0}^{n/2-1} m_a\varphi_a = 0 & \text{for } n \text{ even} \\ \sum\limits_{a=0}^{(n-1)/2-1} 2m_a\varphi_a + m_{(n-1)/2}\varphi_{(n-1)/2} = 0 & \text{for } n \text{ odd} \end{array} \right. \tag{2.2}$$

with $2\sum\limits_{a=0}^{n/2-1} m_a = m$ (for even n) and by $2\sum\limits_{a=0}^{(n-1)/2-1} m_a + m_{(n-1)/2} = m$ (for odd n)[11]. The cases with $m_a = 1, a = 0, \cdots, E(\frac{n-1}{2})$ correspond to particular mus-functions, called 'minimal" in the sequel.

3 Duality in multisine-Gordon models

Without loss of generality the multisine-Gordon model $MSG(n|m)$ for $d=2$ is defined [7, 10, 11] in terms of the function mus_0. The action writes

$$\mathcal{A}^{(n|m)} = \int d^2x \Big[\frac{1}{16\pi} \big[\sum_{a=0}^{E(n/2)-1} (\partial_\mu \phi_a)^2 + \sum_{a=0}^{n-E(n/2)-2} (\partial_\mu \varphi_a)^2 \big] - \mu \mathrm{mus}_0(\{\alpha_a \phi_a\}, \{\beta_a \varphi_a\}) \Big], \quad (3.1)$$

with $E(n)$ the integer part of n, μ real, (α_a,β_a) complex parameters and $(n|m)$ stands for the "embedding" considered. This model is denoted as $MSG_{(n|m)}(\{\alpha_a\}, \{\beta_a\})$.

Following the approach of A. Zamolodchikov [4], quantum integrability in $2d$ is related to the conservation of a non-trivial current (spin higher than two). The analyzis can be carried out from massless perturbation theory in which the whole exponential terms are treated as interaction terms, with tadpoles divergences removed by normal ordering [1]. A conjecture is made for local holomorphic currents in terms of a sum of all possible operators $(\partial^p \Phi_a)^{n_1} (\partial^q \Phi_b)^{n_2}$ $(\Phi = (\phi, \varphi))$ of a given overall spin s compatible with the symmetries of the lagrangian and with specific coefficients $X^{n_1,n_2}_{a,p,b,q}$. An operator product expansion (OPE) is then performed for the action of the current on the operator vertices. In conformal perturbation theory (CPT) the conservation of the current to first order in the interaction term requires the necessary vanishing of all terms which cannot be expressed as exact derivatives. A set of linear equations results for the coefficients $X^{n_1,n_2}_{a,p,b,q}$. For the multisine-Gordon models this system has three types of solutions leading to constraints on the parameters α_a and β_a. Solutions corresponding to particular constant values of α_a and β_a are exotic and will not be analyzed. The second type is characterized by a mixing of real and complex values of the parameters satisfying a quadratic equation. Some of these cases correspond to Affine Toda Field Theories (ATFT) discussed in the next section. The last type is given by *real* values of the parameters subject to quadratic constraints.

For these real values the solvability conditions correspond to an hypersurface in the parameter space $\{\alpha_a, \beta_a\}$ which turns out to have in general a simple geometrical interpretation. As we shall see the embedding property of the multicomplex space permits now to take into account known perturbed CFT and leads to new models which could not be tackled in [7]. This family of models are not only specified by n and $m_a, a = 0, \cdots E(\frac{n-1}{2})$, but also by the relation(s) among the parameters α_a and β_a.

The type of solutions given by *real* values of the parameters subject to quadratic constraints and showing interesting duality properties is now discussed.

For $n=3$ the current of spin 4 is conserved provided the parameters obey the constraints

$$\begin{array}{lll} (i) & \alpha_0^2 - \beta_0^2 = 1/2, & m_0/m_1 = 1, 1/2 \\ (ii) & \frac{m_0}{m_1}\big[\alpha_0^2 - \beta_0^2\big] - \frac{m_0}{m_1}\big[2\frac{m_0}{m_1}\beta_0^2\big] - \big[\frac{1}{2} + \frac{m_0}{m_1}\big] = 0, & m_0, m_1 \neq 0. \end{array} \quad (3.2)$$

The case (i) with $m_0/m_1 = 1$, (minimal representation) was already envisaged in [7]. Its integrability and dual properties were derived in [6], but outside the present general context.

[1]The whole set of parameters is assumed to satisfy the renormalizibility condition *i.e.* no other vertex operator terms are generated under RG group flow; then conservation to first order in the interaction term is sufficient to ensure integrability.

The case with $m_0/m_1 = 1/2$ corresponds to the embedding $I\!\!M\!\!C_3 \subset I\!\!M\!\!C_4$. Its duality property will be discussed below.

For $n = 4$ a spin 4 current is conserved if

$$
\begin{array}{llr}
(i) & \left\{ \begin{array}{l} \alpha_0^2 - \beta_0^2 = 1/2 \\ \alpha_0 = \alpha_1 \end{array} \right. & m_0/m_1 = 1 \\
 & & \\
(ii) & \left\{ \begin{array}{l} \alpha_0^2 - \beta_0^2 = 1/2 \\ \alpha_1^2 - \frac{1}{4\beta_0^2} = 1/2 \end{array} \right. & m_0/m_1 = 1/(2\beta_0^2) \\
 & & \\
(iii) & \frac{m_0}{m_1}\left[\alpha_0^2 - \beta_0^2\right] + \left[\alpha_1^2 - \frac{m_0^2}{m_1^2}\beta_0^2\right] - \left[1 + \frac{m_0}{m_1}\right] = 0, & m_0, m_1 \neq 0.
\end{array}
\tag{3.3}
$$

The first solution has properties similar to the earlier case (i) [6]. Solution (ii) has a self-dual representation for which the general discussion of the next section is relevant.

Finally, for $n = 5$ only the generic solutions and one already studied in [6] (of type (i)-Eq.(3.2) or (i)-Eq.(3.3)) will be discussed. The conditions are respectively

$$
\begin{array}{ll}
(i) & \alpha_0^2 - \beta_0^2 = 1/2,\ \alpha_0 = \alpha_1, \beta_0 = -\beta_1, \qquad\qquad m_0/m_2 = m_1/m_2 = 1/2 \\
 & \\
(ii) & \frac{m_0}{m_2}\left[\alpha_0^2 - \beta_0^2\right] + \frac{m_1}{m_2}\left[\alpha_1^2 - \beta_1^2\right] \\
 & -\frac{m_0}{m_2}\left[2\frac{m_0}{m_2}\beta_0^2\right] - \frac{m_1}{m_2}\left[2\frac{m_1}{m_2}\beta_1^2\right] - \left[\frac{1}{2} + \frac{m_0}{m_2} + \frac{m_1}{m_2}\right] = 0, \quad m_0, m_1, m_2 \neq 0.
\end{array}
\tag{3.4}
$$

The first one corresponds to $I\!\!M\!\!C_5 \subset I\!\!M\!\!C_6$. This QFT possesses a dual representation discussed below.

With these constraints the current of spin 4 is conserved to first order in CPT. This is enough to ensure integrability under certain conditions (*cf* footnote 1). The models listed in the table below have this property.

The families of QFT's associated to the conditions (i) in (3.2) and (i) in (3.3) have a simple interpretation. In these two cases, in the action (3.1) one may always consider the terms in $\exp(\beta_0\varphi_0)\cos(\alpha_0\phi_0)$ in $\mathrm{mus}_0\,(\Phi)$ together with the kinetic contributions and with the condition $\alpha_0^2 - \beta_0^2 = 1/2$ as an initial conformal field theory (CFT) deformed by the remaining part of the $\mathrm{mus}_0\,(\Phi)$.

For an appropriate comparison with known results a new normalization is now retained such that ($\alpha_a \rightarrow \alpha_a/\sqrt{8\pi}$, $\beta_a \rightarrow \beta_a/\sqrt{8\pi}$). For small parameters, it is possible to fermionize the QFT's using the $2d$ boson-fermion correspondence [3, 6]. To obtain a well-defined QFT counterterm(s) is (are) added to cancel the fermion loop divergence. Consider firstly the case $n = 3$. For the "minimal" $I\!\!M\!\!C_3$ ($\frac{m_0}{m_1} = 1$, with constraint (i) in (3.2)), after fermionization a massive Thirring model coupled to an $A_1^{(1)}$ ATFT is obtained. Its dual representation is given in [6]. Consider next the MSG model associated to $I\!\!M\!\!C_3 \subset I\!\!M\!\!C_4$ ($\frac{m_0}{m_1} = 1/2$, case (i) in (3.2)). For small $\beta_0 = \beta$, the $2d$ fermion-boson correspondence [3, 6] gives :

$$
\begin{aligned}
\mathcal{A}^{(3|4)} \;=\; & \int d^2x \Big[i\overline{\psi}\gamma_\mu\partial_\mu\psi - \frac{g_0}{2}(\overline{\psi}\gamma_\mu\psi)^2 - M\overline{\psi}\psi e^{\beta\varphi_0} \\
& + \frac{1}{2}(\partial_\mu\varphi_0)^2 - \frac{M^2}{2\beta^2}\big(2e^{-\beta\varphi_0} + e^{2\beta\varphi_0}\big)\Big],
\end{aligned}
\tag{3.5}
$$

with g_0 defined by $g_0/\pi = \frac{4\pi}{\alpha_0^2} - 1$. In the non-minimal case ($\mathbb{MC}_3 \subset \mathbb{MC}_4$), an $A_2^{(2)}$ ATFT (Bullough-Dodd model) is obtained for the purely bosonic part [2]. V. Fateev has shown [6] that the QFT (3.5) possesses a dual representation associated to the complex sinh-Gordon model (CSG) (sigma model with Witten's black hole metric). In this dual representation (with the complex scalar field $\chi = \chi_1 + i\chi_2$), the action is

$$\tilde{\mathcal{A}}^{(3|4)} = \int d^2x \, \left(\frac{1}{2} \frac{\partial_\mu \overline{\chi} \partial_\mu \chi}{1 + (\frac{\gamma}{2})^2 |\chi|^2} - \frac{M^2}{2} |\chi|^2 \Big[1 + (\frac{\gamma}{2})^2 |\chi|^2 \Big] \right) . \tag{3.6}$$

It was denoted $BC_0(\chi, \gamma)$ where the coupling constant γ is defined by $\gamma = \frac{4\pi}{\beta}$.

The minimal $\mathbb{MC}_4$ model ($\frac{m_0}{m_1} = 1$, case (i) in (3.3)) was already envisaged in [7] and its fermionization corresponds to two massive Thirring models coupled to an $A_1^{(1)}$ ATFT. The dual representation is given in [6].

Consider next the MSG model associated to $\mathbb{MC}_5 \subset \mathbb{MC}_6$ for $\alpha_0 = \alpha_1 = \alpha$ and $\beta_0 = -\beta_1 = \beta$, $\frac{m_0}{m_2} = \frac{m_1}{m_2} = \frac{1}{2}$, case (i) in (3.4). For small β, the $2d$ fermion-boson correspondence gives ($g_0 = g_1 = g$) [3] :

$$\begin{aligned}
\mathcal{A}^{(5|6)} \quad &= \quad \int d^2x \Big[\frac{1}{2}(\partial_\mu \varphi_0)^2 + \frac{1}{2}(\partial_\mu \varphi_1)^2 + \sum_{a=0}^{1} \big[i\overline{\psi}_a \gamma_\mu \partial^\mu \psi_a - \frac{g}{2}(\overline{\psi}_a \gamma_\mu \psi_a)^2 \big], \\
&\qquad - M\overline{\psi}_0 \psi_0 e^{\beta\varphi_0} - M\overline{\psi}_1 \psi_1 e^{-\beta\varphi_1} - \frac{M^2}{2\beta^2}\big(e^{2\beta\varphi_0} + 2e^{-\beta(\varphi_0 - \varphi_1)} + e^{-2\beta\varphi_1} \big) \Big],
\end{aligned} \tag{3.7}$$

which corresponds to two massive Thirring (MT) models coupled with a $C_2^{(1)}$ ATFT. It was already shown by V. Fateev [6] by perturbative and non-perturbative analysis that the QFT (3.7) possesses a dual representation $\tilde{\mathcal{A}}^{(5|6)}$

$$\tilde{\mathcal{A}}^{(5|6)} = \int d^2x \, \left(\sum_{a=0,1} \frac{1}{2} \frac{\partial_\mu \overline{\chi_a} \partial_\mu \chi_a}{1 + (\frac{\gamma}{2})^2 |\chi_a|^2} - \frac{M^2}{2} \Big[\chi_0 \overline{\chi_0} + \chi_1 \overline{\chi_1} + (\frac{\gamma}{2})^2 (\chi_0 \overline{\chi_0})(\chi_1 \overline{\chi_1}) \Big] \right) , \tag{3.8}$$

denoted $D_0^{(2)}(\chi_{0,1}, \gamma)$ with γ defined as above.
From the above results and those obtained in [7], the correspondence between known QFTs and MSG theories may be summarized in the following table where, for each integrable MSG, its QFT in terms of MT coupled with specific ATFT and its dual QFT is indicated. In this compilation we only quote MSG models with a known dual representation after fernionization. The important fact is that all these models are generated by MC algebras.

[2] Up to the choice $\mu = \frac{3}{4\beta}(2M)^{\frac{3}{2}}$ and the shift $\varphi_0 \rightarrow \varphi_0 - \frac{1}{\beta}\ln(\frac{\sqrt{2M}}{\beta})$.

[3] For the choice $\mu = \frac{5}{2}(\frac{M^4}{\beta^2})^{\frac{1}{3}}$ and the shifts $\varphi_0 \rightarrow \varphi_0 - \frac{1}{3\beta}\ln(\frac{M}{\beta^2})$ and $\varphi_1 \rightarrow \varphi_1 + \frac{1}{3\beta}\ln(\frac{M}{\beta^2})$, then the purely bosonic part possesses a stable classical vacuum.

n	Multicomplex Embedding	QFT (coupling β)		Dual QFT (coupling $\gamma = \frac{4\pi}{\beta}$)
3	$\mathbb{MC}_3$	MT coupled with $A_1^{(1)}$	(3|3)	CSG
	$\mathbb{MC}_3 \subset \mathbb{MC}_4$	MT coupled with $A_2^{(2)}$	(3|4)	CSG $BC_0(\chi, \gamma)$
4	$\mathbb{MC}_4$	$MT \otimes MT$ coupled with $A_1^{(1)}$	(4|4)	$N = 2$ supersymmetric sine-Gordon,...
5	$\mathbb{MC}_5 \subset \mathbb{MC}_6$	$MT \otimes MT$ coupled with $C_2^{(1)}$	(5|6)	CSG $D_0^{(2)}(\chi_{0,1}, \gamma)$

Clearly the analysis can be carried out for any order n of the multicomplex space, for any set of $\{m_a\}$, and for a specific choice of the constraints on parameters. It is always possible to fermionize the corresponding MSG model as before and to obtain its QFT interpretation in terms of massive Thirrings models. For example the minimal case $\mathbb{MC}_5$ corresponds to two massive Thirring models coupled with a $D_3^{(2)}$ ATFT. Its dual counterpart is an open question. Hence this compilation is not exhaustive since integrable MSG-models may exist with or without a dual representation. The benefit of the embedding property $\mathbb{MC}_n \subset \mathbb{MC}_m$ is that integrable models reached from order n can also be related to MC-numbers of order $m > n$.

4 Comments, perspectives and conclusions

For $n < 6$ different integrable models were found. Their integrability is ensured from different constraints among the parameters. However when $n \geq 6$ the spin 4 current is conserved (to first order in the interaction term) only for the generic solutions (case (ii) in (2.2), (iii) in (2.3) and (ii) in (2.4)). Similar conclusions are found in [13] using different techniques. It can be checked explicitly that all the conditions for current conservation (to first order) of the generic model (m_a not fixed) related to the "embedding" $\mathbb{MC}_n \subset \mathbb{MC}_m$ studied so far are given by a single formula ($m_a \neq 0$)

$$\begin{cases} \sum_{a=0}^{n/2-2} \frac{m_a}{m_{\frac{n}{2}-1}} (\alpha_a^2 - \beta_a^2) + \alpha_{n/2-1}^2 - \sum_{a=0}^{n/2-2} (\frac{m_a}{m_{\frac{n}{2}-1}} \beta_a)^2 - \Big[1 + \sum_{a=0}^{n/2-2} \frac{m_a}{m_{\frac{n}{2}-1}}\Big] = 0, \quad n \text{ even} \\ \sum_{a=0}^{(n-1)/2-1} \frac{2m_a}{m_{\frac{n-1}{2}}} (\alpha_a^2 - \beta_a^2) - \sum_{a=0}^{(n-1)/2-1} (\frac{2m_a}{m_{\frac{n-1}{2}}} \beta_a)^2 - \Big[1 + \sum_{a=0}^{(n-1)/2-1} \frac{2m_a}{m_{\frac{n-1}{2}}}\Big] = 0, \; n \text{ odd.} \end{cases} \tag{4.1}$$

When the parameters are real and after a proper rescaling of the α_a's and β_a's the geometric interpretation is simply that current conservation at least to the first order occurs if the parameters live on hyperboloids. They are invariant under a pseudo-rotational group [11].

When imaginary parameters are allowed some MSG-models possess interesting dual properties in terms of coupling. Using the flexibility of the non-minimal and non-embedded representation of the multisine it is possible to obtain a dual counterpart of a given MSG. Consider for instance the MSG-model associated with $n = 4$. Any choice (integer or not) for m_0 and m_1 is possible when relaxing the embedding interpretation. For the set

$$\left\{\begin{array}{l}\alpha_0^2-\beta_0^2=\frac{1}{2}\\ \alpha_1=\frac{i}{2\beta_0}\end{array}\right. \qquad \frac{m_0}{m_1}=\frac{1}{2\beta_0^2} \tag{4.2}$$

the spin 4 current

$$\begin{aligned}
T^{(4|m)} \;=\;& \frac{3+10\beta_0^2}{12\beta_0^2}\;(\partial\phi_0)^4+\frac{\beta_0^2}{3+6\beta_0^2}\;(\partial\phi_1)^4+\frac{\beta_0^2}{3+6\beta_0^2}\;(\partial\varphi_0)^4\\
+\;& \Big[(\partial\phi_0)^2(\partial\varphi_0)^2+(\partial\phi_1)^2(\partial\varphi_0)^2+(\partial\phi_0)^2(\partial\phi_1)^2\Big]\\
+\;& \frac{2+4\beta_0^2}{\beta_0}\;(\partial\phi_0)^2(\partial^2\varphi_0)-\frac{1}{\beta_0}\;(\partial\phi_1)^2(\partial^2\varphi_0)\\
+\;& \frac{7+24\beta_0^2+16\beta_0^4}{3+6\beta_0^2}\;(\partial^2\varphi_0)^2+\frac{3+22\beta_0^2+56\beta_0^4+32\beta_0^6}{6\beta_0^2+12\beta_0^4}\;(\partial^2\phi_0)^2\\
+\;& \frac{3+19\beta_0^2+36\beta_0^4+16\beta_0^6}{3\beta_0^2+6\beta_0^4}\;(\partial^2\phi_1)^2.
\end{aligned} \tag{4.3}$$

is conserved. For the set

$$\left\{\begin{array}{l}\alpha_0=-i\beta_0\\ \alpha_1^2-\frac{1}{4\beta_0^2}=\frac{1}{2}\end{array}\right. \qquad \frac{m_0}{m_1}=\frac{1}{2\beta_0^2} \tag{4.4}$$

a spin 4 current is also conserved. It is given by Eq.(4.3) with the duality transformation $\sqrt{2}\beta_0 \longrightarrow -\frac{1}{\sqrt{2}\beta_0}$ together with the change $\phi_0 \leftrightarrow \phi_1$. This model is integrable and possesses a dual representation in terms of complex sinh-Gordon model coupled to a massive Thirring [14]. Its alternative truly bosonic representation is obtained from the initial lagrangian with the change (4.4). When $\sqrt{2}\beta_0 \longrightarrow -\frac{1}{\sqrt{2}\beta_0}$, $m_0/m_1 \longrightarrow 2\beta_0^2$, it is always possible to find new parameters m'_a such that $m'_0/m'_1 = 1/(2\beta_0^2)$. A new generator $e^\vee$ (the "dual" of e) of the multicomplex space is then obtained from the initial one e by the exchange $m_0 \leftrightarrow m_1$, which corresponds in terms of coupling to the exchange $\sqrt{2}\beta_0 \longrightarrow -\frac{1}{\sqrt{2}\beta_0}$. In other words, the duality transformation (4.4) induces a transformation among the m_a's themselves, that is a duality transformation in the MC-algebras.

One other legitimate question is whether or not *any* ATFT potential can be described within an MSG-model. A multisine-Gordon model of order n is specified by the set of m_a's and the parameters $\{\alpha_a, \beta_a\}$. In the previous section, the parameters α_a, β_a were *real*. However when $\alpha_a \in I\!R$ and $\beta_a \in iI\!R$, different ATFT potentials can be reached depending on the sets of $\{m_a\}$ and/or $\{\alpha_a, \beta_a\}$. Since in particular for $n=3$ the affine Lie algebra $A_2^{(1)}$ and $D_3^{(2)}$ can be obtained for imaginary parameters as $MSG_{(3|3)}(\beta\sqrt{\frac{3}{2}}, i\frac{\beta}{\sqrt{2}})$ and $MSG_{(3|3)}(\beta, i\beta)$ the first step is to consider the so-called totally compact mus-functions based on a complex algebra. Then one way to proceed is through the identification of the vertex operators associated to Lie algebras with the corresponding operators obtained from the expansion of $\mathrm{mus}_0(\Phi)$ [4] In the process the

[4]An alternative is to use the approach on non-local conserved charges [13].

condition $\sum_{a=0}^{n-1} n_a r_a = 0$, where $\{n_a\}$ are the Kač's labels and $\{r_a\}$ the roots of the affine Lie algebra, is in correspondence with the condition of unimodularity Eqs. (2.2). However some care is needed within this identification. On the one hand the metric of the ATFT governs the identification of the induced metric of the MSG. It is in general non-invariant. On the other hand the constraint among the MSG-fields depends in an essential way on the manner the Lie algebra condition $\sum_{a=0}^{n-1} n_a r_a = 0$ is implemented. The final outcome is that *all* ATFT potentials can be related to an MSG one. The detailed analysis is in perfect agreement with other results [13].

To conclude a class of new multi-parameter models has been found with a generic condition for current conservation to first order in the interaction term. In order to ensure integrability of these new models, it remains to check that quantum corrections do not not spoil current conservation. Finding a general method to derive their dual counterparts is a challenging problem.

A field theoretic formulation in terms of minimal and non-minimal representation of MC-algebra and multisine functions provides a unified algebraic description of various integrable $2d$ deformed Toda field theories. Thereby their studies is reduced to that of the underlying multi-paramater space. Some additional indications were thus gathered on the richness of the algebraic structure of MSG in the understanding of hidden symmetries which may appear in multi-parametrized quantum field theories.

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